ISRAEL JOURNAL OF MATHEMATICS, Vol. 20, Nos. 3–4, 1975

ON THE LOCAL STRUCTURE OF SUBSPACES OF BANACH LATTICES

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ABSTRACT

The conjecture that every Banach space contains uniformly complemented l_p^n's for some $1 \leqq p \leqq \infty$ is verified for subspaces of Banach lattices which do not contain l_∞^n's uniformly.

I. One of the main problems in the study of local (i.e., finite dimensional) structure of Banach spaces is whether Lindenstrauss' [7] "uniformly complemented l_p^n-conjecture" is true; i.e., whether every infinite dimensional Banach space X contains a uniformly complemented sequence (E_n) of subspaces such that, for some p, $1 \leqq p \leqq \infty$, $\sup d(E_n, l_p^n) < \infty$. (Here $d(E,F)$ is the Banach-Mazur distance coefficient $\inf\{\|T\|\cdot\|T^{-1}\| : T$ is an isomorphism from E onto $F\}$.) Recently the second-named author verified the uniformly complemented l_p^n-conjecture for Banach spaces X which have an unconditional basis (cf. [15]). It follows from the main result in the present paper that the uniformly complemented l_p^n-conjecture is also true for Banach spaces which are Banach lattices or, more generally, have local unconditional structure. Our main result, combined with the results in [5], in fact yields the following:

THEOREM 1. *Suppose that X is a subspace of a Banach lattice L and L does not contain l_∞^n uniformly for all n.*

A. *Given $K < \infty$, $\varepsilon > 0$ and an integer n, there is an integer $N = N(K, \varepsilon, L)$ so that if $E \subseteq X$ and $d(E, l_1^N) \leq K$, then $E \supseteq F$ with $d(F, l_1^n) \leqq 1 + \varepsilon$ and F is $1 + \varepsilon$-complemented in L. Moreover, if X is not super-reflexive, then X contains l_1^n uniformly for large n.*

B. *If X is super-reflexive, then given any sequence (H_n) of subspaces of X with $\dim H_n \to \infty$, there is a uniformly complemented sequence of subspaces (G_n) with $G_n \subseteq H_n$, $d(G_n, l_2^{k(n)}) \leq 2$, and $k(n) \to \infty$.*

[†] Supported by NSF Grant GP-33578.
Received September 6, 1974

Part A of Theorem 1 was proved in [5]. After this paper was submitted, we realized in conversations with Pelczynski and Rosenthal that by using the argument for case B and considerably more machinery, it can be shown in case B that X is locally π-Euclidean in the sense of Pelczynski and Rosenthal [12]; i.e., there is a $\lambda < \infty$ so that for each integer n, there is $N = N(n)$ such that if $E \subseteq X$ and $\dim E \geqq N(n)$ then $E \supseteq F$ with $d(F, l_2^n) < 2$ and F is λ-complemented in X.

Part B gives a wide class of super-reflexive spaces which are sufficiently Euclidean in the sense of Stegall and Retherford [14]; i.e., which contain uniformly complemented copies of l_2^n for all n. It should be noted that super-reflexivity cannot be replaced by reflexivity in Part B, since in [6] a reflexive Banach space with unconditional basis is constructed which does not contain l_∞^n uniformly for all n but which is not sufficiently Euclidean.

As an immediate corollary of Theorem 1, we have

COROLLARY 1. *If X is a subspace of a Banach lattice which does not contain l_∞^n uniformly for all n, then X contains uniformly complemented subspaces E_n with* $\sup d(E_n, l_1^n) < \infty$ or $\sup d(E_n, l_2^n) < \infty$.

We use standard Banach space theory notation as may be found in [9]. For definitions of not yet standard terms (e.g., local unconditional structure) see [5].

II. The main result

Throughout this section we assume that X is a subspace of a Banach lattice L for which there is $q < \infty$ so that if $x, y \in L$ with $|x| \wedge |y| = 0$, then $\|x + y\|^{q'} \geqq \|x\|^q + \|y\|^q$, i.e. L admits *a lower l_q estimate.* It was observed in [5] that if L is any Banach lattice which does not contain l_∞^n uniformly for large n, then there is $q < \infty$ and an equivalent lattice norm on L which satisfies such a lower l_q estimate. Indeed, by Corollary III.4 in [5] or Maurey's generalization [10] of a result of Rosenthal [13], if $(L, |||\cdot|||)$ is a Banach lattice which does not contain l_∞^n uniformly for all n, there exist $c > 0$ and $q < \infty$ so that $|||\, \Sigma x_i \,|||^q \geqq c(\Sigma\, |||\, x_i \,|||^q)$ whenever $(x_i) \subseteq L$ with $|x_i| \wedge |x_j| = 0$ for $i \neq j$. Define $\|\cdot\|$ on L by $\|x\| = \sup\{(\Sigma\, |||\, x_i \,|||^q)^{1/q} : |x_i| \wedge |x_j| = 0$ for $i \neq j$ and $\Sigma x_i = x\}$; it is easy to check that $\|\cdot\|$ is an equivalent Banach lattice norm on L which satisfies a lower l_q estimate.

We would like to recall some observations, most of which are due to Meyer-Nieberg [11]. Given a non-negative functional x^* on L with $\|x^*\| = 1$, one can define a semi-norm $\|\cdot\|_{x^*}$ on L by setting $\|x\|_{x^*} = x^*(|x|)$. It is clear that $\|\cdot\|_{x^*}$ is additive on the positive cone of L, hence (after dividing out by the ideal $\{x : x^*(|x|) = 0\}$) the completion of $(L, \|\cdot\|_{x^*})$ is an abstract L-space which is isometric to $L_1(\mu)$ for some measure μ, by Kakutani's well-known theorem. Obviously, the natural identity mapping from $(L, \|\cdot\|)$ to $(L, \|\cdot\|_{x^*})$ is norm decreasing. Now since $(L, \|\cdot\|)$ satisfies a lower l_q estimate, L is σ-complete and σ-order continuous, so that in fact the image of L in $L_1(\mu)$ is an order ideal.

Given a finite dimensional subspace H of L, we will be interested in the behavior of the ratio between $\|x\|$ and $\|x\|_{x^*}$ as x ranges over H and x^* over nonnegative functionals of unit norm. We let

$$\delta(H) = \sup_{x^* \geq 0; \|x^*\| = 1} \inf \{x^*(|x|) : x \in H, \|x\| = 1\}.$$

Our first main step in the proof of Part B in Theorem 1 will be to show that if $\delta(H)$ is small, then H contains a "long" finite sequence of vectors which are almost disjoint. More precisely, we have

LEMMA 1. *Given k there is $0 < \epsilon = \epsilon(k,q)$ so that if H is a subpace of L and $\delta(H) < \epsilon$, then there are unit vectors $(x_i)_{i=1}^k$ in H and disjoint vectors $(y_i)_{i=1}^k$ in L so that $0 \leqq |y_i| \leqq |x_i|$ and $\|x_i - y_i\| \leqq (2k)^{-1}$ for $1 \leqq i \leqq k$.*

PROOF. The argument used here was suggested by the proof of Proposition 3.1 in [12]. Choose $\epsilon = \epsilon(k,q)$ so that $[1-(1-2k^3\epsilon)^q]^{1/q} < 2^{-1}k^{-2}$. If $\delta(H) < \epsilon$, we show that $(x_i)_{i=1}^k$ and $(y_i^h)_{h=1}^k{}_{i=1}^h$ can be constructed by induction so that for each j, $1 \leqq j \leqq k$, $(y_i^j)_{i=1}^j$ are disjoint, $0 \leqq |y_i^j| \leqq |x_i|$, and

$$\|x_1 - y_i^j\| < (j-1)2^{-1}k^{-2}. \tag{1}$$

After showing this we will set $y_i^k = y_i$ thus completing the proof. For the first step, simply choose a unit vector x_1 in H and set $y_1^1 = x_1$. Suppose that $(x_i)_{i=1}^j$ and $(y_i^j)_{i=1}^j$ have been defined to satisfy (1). For $1 \leqq i \leqq j$, choose non-negative $y_i^* \in L^*$ so that $\|y_i^*\| = 1$ and $y_i^*(|y_i^j|) = \|y_i^j\|$. Of necessity, $y_i^*(y_h^j) = 0$ for $i \neq h$. Let $x^* = (\Sigma_{i=1}^j y_i^*) \cdot \|\Sigma_{i=1}^j y_i^*\|^{-1}$. By the definition of $\delta(H)$, there is a unit vector x_{j+1} in H, for which $x^*(|x_{j+1}|) < \epsilon$.

Let $u = (|x_{j+1}| - 2^{-1}k^{-2}\Sigma_{i=1}^j |y_i^j|)^+$.

(If L is a function lattice, the support of u is the set where $|x_{j+1}|$ is larger than a small multiple of $\Sigma_{i=1}^j |y_i^j|$). Let P be the band projection on L generated by u (so that if L is a function lattice, P is multiplication by the characteristic function of supp u, and, in general lattices, $Py = \vee_{n=1}^\infty (nu \wedge y^+) - \vee_{n=1}^\infty (nu \wedge y^-)$ for $y \in L$). Set $y_{j+1}^{j+1} = Px_{j+1}$. It is clear that condition (1) holds for $\|x_{j+1} - y_{j+1}^{j+1}\|$.

Indeed, one obtains from the definition of u that $\|x_{j+1}-Px_{j+1}\| \leqq 2^{-1}k^{-2}\|\sum_{i=1}^{j}|y_i^j|\| \leqq 2^{-1}k^{-2}j$.

Set $y_i^{j+1}=y_i^j-Py_i^j$ for $1 \leqq i \leqq j$. Then obviously $|y_i^{j+1}|\wedge|y_h^{j+1}|=0$ for $1 \leqq i < h \leqq j+1$ and $0 \leqq |y_i^{j+1}| \leqq |x_i|$ for $1 \leqq i \leqq j+1$. Hence the proof will be complete if we can show that $\|Py_i^j\|=\|y_i^j-y_i^{j+1}\|<2^{-1}k^{-2}$. Observe that $|Py_i^j|=P|y_i^j| \leqq P(\sum_{i=1}^{j}|y_i^j|) \leqq 2k^2|x_{j+1}|$, and hence $y_i^*(|Py_i^j|) \leqq jx^*(|Py_i^j|) \leqq 2jk^2x^*(|x_{j+1}|)<2k^3\epsilon$ whence

$$\|y_i^j-Py_i^j\| \geqq y_i^*(|y_i^j-Py_i^j|) \geqq \|y_i^j\|-2k^3\epsilon.$$

Since L satisfies a lower l_q estimate, we have that

$$\|Py_i^j\|^q \leqq \|y_i^j\|^q-\|y_i^j-Py_i^j\|^q$$

so that

$$\|Py_i^j\| \leqq [\|y_i^j\|^q-(\|y_i^j\|-2k^3\epsilon)^q]^{1/q} \leqq [1-(1-2k^3\epsilon)^q]^{1/q} \leqq 2^{-1}k^{-2}$$

as desired. This completes the proof.

Before stating the next lemma, let us recall some notation. Given a system of vectors $(x_i)_{i=1}^{2^k}$ in a Banach space, the Rademacher elements $r_1,\cdots,r_k$ over (x_i) are defined by

$$r_1=\sum_{i=1}^{2^k}x_i;$$

$$r_2=\sum_{i=1}^{2^{k-1}}x_1-\sum_{i=2^{k-1}+1}^{2^k}x_i;$$

$$r_3=\sum_{i=1}^{2^{k-2}}x_i-\sum_{i=2^{k-2}+1}^{2^{k-1}}x_i+\sum_{i=2^{k-1}+1}^{3\cdot 2^{k-2}}x_i-\sum_{i=3\cdot 2^{k-2}+1}^{2^k}x_i,\cdots$$

$$r_k=x_1-x_2+x_3-x_4\cdots-x_{2k}.$$

Observe that if the x_i's are disjoint vectors in a Banach lattice, then all the Rademacher elements over the x_i's have the same norm.

LEMMA 2. *There is a constant $c=c(q)<\infty$ so that if $(x_i)_{i=1}^{2^k}$ are disjoint unit vectors in L (which is assumed to have a lower l_q estimate) and $2^{-1} \leqq (\|\sum_{i=1}^{m}a_ix_{h_i}\|/\|\sum_{i=1}^{m}a_ix_i\|) \leqq 2$ for every $h_1<h_2<\cdots<h_m$ and every set of scalars $\{a_i\}$, then the normalized Rademacher elements $(\|r_1\|^{-1}\cdot r_i)_{i=1}^{k}$ are*

c-equivalent to the unit vector basis of l_2^k. *Further, there is* $x^* \geqq 0$ *in* L^* *with* $\|x^*\| = 1$ *for which* $\|x\|_{x^*} = x^*(|x|) \geqq (12c)^{-1}\|x\|$ *for each* $x \in [r_i]_{i=1}^k$.

PROOF. The first conclusion was verified in [15]; the constant c depends only on the constants in Khintchine's inequalities for $L_1[0,1]$ and $L_q[0,1]$.

Using a space isometry of L, we can assume that $x_i \geqq 0$ for $1 \leqq i \leqq 2^k$. Indeed, if P_i is the band projection on L generated by x_i^-, then $Sx = x - 2\Sigma_{i=1}^{2^k} P_i x$ defines an isometry on L so that $Sx_i \geqq 0$ for $1 \leqq i \leqq 2^k$ and $|Sx| = |x|$ for $x \in L$.

Let z^* be the functional in the dual of $[x_i]_{i=1}^{2^k}$ defined by

$$z^*\left(\sum_{i=1}^{2^k} c_i x_i\right) = 2^{-k}\left(\sum_{i=1}^{2^k} c_i\right)\cdot\left\|\sum_{i=1}^{2^k} x_i\right\|.$$

In view of our assumptions on the system $\{x_i\}_{i=1}^{2^k}$ it is easy to check that $1 \leqq \|z^*\| \leqq 4$. Let now x^* be any non-negative Hahn-Banach extension of $\|z^*\|^{-1}\cdot z^*$ to an element of L^*. Then

$$x^*(x_i) = \|z^*\|^{-1}\cdot 2^{-k}\left\|\sum_{i=1}^{2^k} x_i\right\|$$

$$\text{i.e.} \qquad 4^{-1}\cdot 2^{-k}\left\|\sum_{i=1}^{2^k} x_i\right\| \leqq x^*(x_i) \leqq 2^{-k}\left\|\sum_{i=1}^{2^k} x_i\right\|.$$

We also have

$$x^*(|r_j|) = \|z^*\|^{-1}\cdot\left\|\sum_{i=1}^{2^k} x_i\right\|$$

$$\text{i.e.} \qquad 4^{-1}\left\|\sum_{i=1}^{2^k} x_i\right\| \leqq x^*(|r_j|) \leqq \left\|\sum_{i=1}^{2^k} x_i\right\|.$$

Thus, in the abstract L-space $(L, \|\cdot\|_{x^*})$ the x_i's are disjoint vectors having essentially the same norm, so by the classical Khintchine inequality in L_1, we have that

$$\left\|\sum_{j=1}^{k} \alpha_j r_j\right\|_{x^*} \geqq 12^{-1}\left(\sum_{j=1}^{k} |\alpha_j|^2\right)^{1/2}\left\|\sum_{i=1}^{2^k} x_i\right\|.$$

On the other hand, since the normalized Rademacher elements are c-equivalent to the unit vector basis of l_2^k,

$$\left\|\sum_{j=1}^{k} \alpha_j r_j\right\| \leqq c\left(\sum_{j=1}^{k} |\alpha_j|^2\right)^{1/2}\left\|\sum_{i=1}^{2^k} x_i\right\|$$

which means that $\|x\|_{x^*} \geqq (12c)^{-1}\|x\|$ for all $x \in [r_j]_{j=1}^k$.

LEMMA 3. *Assume X is a super-reflexive subspace of L and $c > 0$. Then there is a constant λ so that for every n there is $k(n)$ such that if H is a subspace of X with* $\dim H \geqq k(n)$ *for which there exists $x^* \in L^*$, $x^* \geqq 0$, $\|x^*\| = 1$ with $\|x\|_{x^*} \geqq c\|x\|$ for all $x \in H$, then H contains an n dimensional subspace which is λ-complemented in X.*

PROOF. We need the following generalization by Maurey [10] of a theorem of Rosenthal's [13]:
There are a constant τ and $2 > p > 1$ so that if $T: X \to L_1(\mu)$ is an operator from X into $L_1(\mu)$ for some measure μ, then there are a measure ν and operators $S: X \to L_p(\nu)$, $U: L_p(\nu) \to L_1(\mu)$ for which $US = T$, $\|S\| \leqq \tau\|T\|$, and $\|U\| = 1$.

We also need the fact that $L_p(\nu)$ is, for $2 > p > 1$, locally π-Euclidean. As observed by Pelczynski and Rosenthal [12], this follows from the argument used in [2]. Let τ, p be as given by the Rosenthal-Maurey Theorem and let $k = k(n)$ and M be such that every $k(n)$ dimensional subspace of $L_p[0,1]$ (and hence of $L_p(\nu)$ for arbitrary ν) has an n-dimensional subspace which is M-complemented in L_p.

Suppose that H and x^* are as in the hypothesis. Let $T: (L, \|\cdot\|) \to$ completion of $(L, \|\cdot\|_{x^*})$ be the formal identity and let $X \xrightarrow{S} L_p(\nu) \xrightarrow{U}$ completion of $(L, \|\cdot\|_{x^*})$ be a factorization of T satisfying $\|S\| \leqq \tau$; $\|U\| = 1$. Let P be a projection from $L_p(\nu)$ onto an n-dimensional subspace of SH with $\|P\| \leqq M$. Now $\|Tx\|_{x^*} \geqq c\|x\|$ for $x \in H$, hence (since $\|U\| \leqq 1$), $\|Sx\|_{L_p(\nu)} \geqq c\|x\|$ for $x \in H$. Thus the operator $Q = S^{-1}PS$ is a projection from X onto an n-dimensional subspace of H, and $\|Q\| \leqq c^{-1}M\tau$. This completes the proof.

PROOF OF THEOREM 1, PART B. Of course, by Dvoretzky's theorem we may assume that $d(H_n, l_2^{\dim H_n}) \leqq 2$ for each n. Let $d(n) = \max\{m : H_n$ contains unit vectors $(x_i)_{i=1}^m$ for which there are $(y_i)_{i=1}^m$ in L so that $0 \leqq |y_i| \leqq |x_i|$ and $\|x_i - y_i\| < (2m)^{-1}$ for $1 \leqq i \leqq m\}$. We can decompose (H_n) into two subsequences (one of which may be void) (H_{n_i}) and (H_{m_i}) so that $d(n_i) \to \infty$ and $\sup d(m_i) < \infty$. By Lemma 1, there is $\epsilon > 0$ so that $\inf \delta(H_{m_i}) > \epsilon > 0$. Hence there are $x_i^* \geqq 0$ in L^* with $\|x_i^*\| = 1$ and $\|x\|_{x_i^*} \geqq \epsilon\|x\|$ for all $x \in H_{m_i}$, whence from Lemma 3 we conclude the existence of subspaces $G_{m_i} \subseteq H_{m_i}$ with $\dim G_{m_i} \to \infty$ and G_{m_i} being uniformly complemented in X.

We look now at the H_{n_i}'s. A standard perturbation argument yields the existence of isomorphisms T_i from L onto L with $\|T_i - \text{Identity}\| \leqq \frac{1}{2}$ so that

$T_iH_{n_i}$ contains a sequence of length $d(n_i)$ of disjoint unit vectors. Thus we can assume without loss of generality that H_{n_i} itself contains such a sequence. We use now the following special case of a result of Brunel-Sucheston [1]: Given n, there is $M = M(n)$ so that if $(y_i)_{i=1}^M$ are disjoint unit vectors in a lattice, then there is a subsequence $(x_i)_{i=1}^n$ of $(y_i)_{i=1}^M$ such that

$$2^{-1} \leqq \left(\left\| \sum_{i=1}^m a_i x_{h_i} \right\| \Big/ \left\| \sum_{i=1}^m a_i x_i \right\| \right) \leqq 2$$

$$\text{for every } h_1 < \cdots < h_m$$

and every set of scalars $\{a_i\}$. Since $d(n_i) \to \infty$, we obtain from Lemmas 2 and 3 that $H_{n_i} \supseteq G_{n_i}$ with $\dim G_{n_i} \to \infty$ and (G_{n_i}) uniformly complemented in X.

Remark 1. If X has local unconditional structure, then either X contains (necessarily uniformly complemented) l_∞^n uniformly for all n or, by a theorem from [3], X is isomorphic to a subspace of a Banach lattice L which does not contain l_∞^n uniformly for large n. Thus, by Theorem 1, the uniformly complemented l_∞^n conjecture is true for such an X.

Remark 2. The example of James constructed in [4] does not contain either l_1^n or l_∞^n uniformly for large n and in view of the results of [16], does not isomorphically embed into a Banach lattice which does not contain l_∞^n uniformly for large n. However, it might be true that every super-reflexive Banach space isomorphically embeds into a Banach lattice which does not contain l_∞^n for all n.

Remark 3. The infinite dimensional version of Part B of Theorem 1 is false. In [8], a super-reflexive Banach space with unconditional basis is constructed so that it contains a copy of l_2 but no copy of l_2 is complemented in the space. However, a simpler version of the proof of Theorem 1 and the known fact (cf., e.g. [12] for a proof) that every copy of l_2 in L_p $(1 < p < 2)$ contains an infinite dimensional subspace complemented in L_p yields that if L is a super-reflexive Banach lattice in which no sequence of disjoint vectors is equivalent to the unit vector basis of l_2, then every copy of l_2 in L contains an infinite dimensional subspace which is complemented in L.

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